In summary, the details of the blast wave resulting from the detonation-propulsion system is explained by both calculation and experiments. The relative importance of pressure peaks and valleys on the system performance is illustrated; especially, the importance of the under-pressure is emphasized.

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Stability of a Future Generation Spacecraft Attitude Control System

A. S. C. Sinha*
Purdue University, Indianapolis, Ind.
and
A. Nadkarni†
Old Dominion University, Va.

I. Introduction

THE system under consideration at NASA Langley Research Lab., represents a new concept in attitude control of spacecraft. It will provide a better attitude control, more payload capacity in the spacecraft, and hopefully will meet increasing demands in satellite communication, weather prediction, etc., which need more payload, and unbalanced loading. An accurate modeling of internal energy dissipation is extremely difficult. However, the energy dissipation in either the high-speed Annular Momentum Control Device (AMCD) or the despun main spacecraft structure introduced through a damping mechanism, is modeled by one ball-in-the-tube damper with one degree of freedom.

The subscript of a representation of a vector in a three-dimensional space denotes that vector in a particular coordinate system. For example, (T_{ax}, T_{ay}, T_{az}) and (T_{sx}, T_{sy}, T_{sz}) are the vector representations of bearing interaction torques T_a and T_s in (x,y,z), respectively, where subscript a denotes AMCD and s denotes spacecraft. The projections of a vector T_a in x, y, and z coordinates are denoted by T_{ax}, T_{ay}, and T_{az}. G_{ax}, G_{ay}, G_{az} and G_{sx}, G_{sy}, G_{sz} are external disturbance torques for the AMCD rim and spacecraft due to three balancing masses and one ball-in-tube damper. The vectors ω_a, I_a and ω_s, I_s are angular velocity and transverse moment of inertia of the a coordinate system and the s system, respectively. H_a and H_s are total angular momentum of AMCD and spacecraft, respectively. The Euler angles chosen to represent the positions of the a and s coordinate axes with respect to an inertial axis set are ϕ and θ. α and β are spacecraft rotation angle and rim rotation angle, respectively. P and P' represent damper motion along the z axis. The reader is strongly encouraged to see Ref. 1 for details of the notations and development of equations of motion for dissipative AMCD spacecraft.

II. Stability Criteria

The analysis is based on the dissipative dual-spin configuration. The AMCD spacecraft is comprised of three parts: 1) the primary part of the spacecraft, assumed to be essentially the right circular cylinder; 2) the AMCD spin assembly configuration consists of a rotating rim (no central hub) suspended by noncontacting magnetic bearings and powered by a noncontacting linear electromagnetic motor (see Ref. 2); 3) the energy dissipation in either the high-speed AMCD or the despun main spacecraft structure, introduced through a damping mechanism, is modeled by one ball-in-tube damper. Three particles, each of whose mass is equal to the total mass of the damping mechanism, are rigidly attached to the spacecraft in such a way that the combination damper and three particles become inertially symmetrical about the spin axis.

The equation of motion for the dissipative AMCD-spacecraft can be written as

$$I_a\ddot{\phi}_a + K_\phi(\phi_a - \phi_s) + K_{\dot\phi}(\dot\phi_a - \dot\phi_s) + H_a\dot\theta_a + K_{\dot\phi}\dot\beta(\theta_a - \theta_s) - G_{ax} = 0 \tag{1}$$

$$I_a\ddot{\theta}_a + K_\phi(\theta_a - \theta_s) + K_{\dot\phi}(\dot\theta_a - \dot\theta_s) + H_a\dot\phi_a - K_{\dot\phi}\dot\beta(\phi_a - \phi_s) - G_{ay} = 0 \tag{2}$$

$$I_s\ddot{\phi}_s - K_\phi(\phi_a - \phi_s) - K_{\dot\phi}(\dot\phi_a - \dot\phi_s) + H_s\dot\theta_s - K_{\dot\phi}\dot\beta(\theta_a - \theta_s) - G_{sx} = 0 \tag{3}$$

$$I_s\ddot{\theta}_s - K_\phi(\theta_a - \theta_s) - K_{\dot\phi}(\dot\theta_a - \dot\theta_s) - H_s\dot\phi_s + K_{\dot\phi}\dot\beta(\phi_a - \phi_s) - G_{sy} = 0 \tag{4}$$

$$\dot H_a = G_{az} = I_{az}\ddot\alpha \tag{5}$$

$$H_s = G_{sz} = I_{az}\ddot\beta \tag{6}$$

$$c\dot P + kP + m[\ddot P - \dot\omega_y a + \omega_z\omega_x a - P(\omega_x^2 + \omega_y^2)] = 0 \tag{7}$$

$$c\dot P' + k'P' + m'[\ddot P' - \dot\omega_y' a' + \omega_z'\omega_x' a' - P'(\omega_x'^2 + \omega_y'^2)] = 0 \tag{8}$$

where

$$G_{az} = (-ma\sin\alpha)\ddot P;\; G_{ay} = (ma\cos\alpha)\ddot P \tag{9}$$

$$G_{sx} = (-m'a'\sin\beta)\ddot P';\; G_{sy} = (m'a'\cos\beta)\ddot P' \tag{10}$$

are linearized disturbance torques. The detailed derivation of equations from the first principle are reported in Ref. 1, and are omitted here for the sake of brevity. The stability of the origin is examined. The linearized equation of motion about the equilibrium point gives the following differential equation of motion:

$$\hat M(t)\ddot x(t) + \hat B(t)\dot x(t) + \hat K(t)x(t) = 0 \tag{11}$$

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*Associate Professor, Department of Electrical Engineering, School of Engineering and Technology.
†Research Assistant, Department of Aerospace.

where $x=(\phi_a, \theta_a, \phi_s, \theta_s, P, P')$. $\hat{M}(t)$, $\hat{B}(t)$ and $\hat{K}(t)$ are matrices:

$$\hat{M}(t)=\begin{bmatrix} I_a & 0 & 0 & 0 & a_{15} & 0 \\ 0 & I_a & 0 & 0 & a_{25} & 0 \\ 0 & 0 & I_s & 0 & 0 & a_{36} \\ 0 & 0 & 0 & I_s & 0 & a_{46} \\ a_{51} & a_{52} & 0 & 0 & m & 0 \\ 0 & 0 & a_{63} & a_{64} & 0 & m' \end{bmatrix}$$

where

$$a_{15}=a_{51}=ma\ \sin\alpha, \quad a_{25}=a_{52}=-ma\ \cos\alpha$$

$$a_{36}=a_{63}=m'a'\ \sin\beta, \quad a_{46}=a_{64}=-m'a'\ \cos\beta$$

$$\hat{B}(t)=\begin{bmatrix} k_{\dot\phi} & I_{az}\omega_{\alpha 0} & -k_{\dot\phi} & 0 & 0 & 0 \\ -I_{az}\omega_{\alpha 0} & k_{\dot\phi} & 0 & -k_{\dot\phi} & 0 & 0 \\ -k_{\dot\phi} & 0 & k_{\dot\phi} & I_{sz}\omega_{\beta 0} & 0 & 0 \\ 0 & -k_{\dot\phi} & -I_{sz}\omega_{\beta 0} & k_{\dot\phi} & 0 & 0 \\ 2a\omega_{\alpha 0}m\ \cos\alpha & 2a\omega_{\alpha 0}m\ \sin\alpha & 0 & 0 & c & 0 \\ 0 & 0 & 2a'\omega_{\beta 0}m'\ \cos\beta & 2a'\omega_{\beta 0}m'\ \sin\beta & 0 & c' \end{bmatrix}$$

$$\hat{K}(t)=\begin{bmatrix} k_\phi & k_{\dot\phi}\omega_{\beta 0} & -k_\phi & k_{\dot\phi}\omega_{\beta 0} & ma\omega_{\alpha 0}^2\sin\alpha & 0 \\ -k_{\dot\phi}\omega_{\beta 0} & k_\phi & k_{\dot\phi}\omega_{\beta 0} & -k_\phi & -ma\omega_{\alpha 0}^2\cos\alpha & 0 \\ -k_\phi & -k_{\dot\phi}\omega_{\beta 0} & k_\phi & k_{\dot\phi}\omega_{\beta 0} & 0 & m'a'\omega_{\beta 0}^2\sin\beta \\ k_{\dot\phi}\omega_{\beta 0} & -k_\phi & k_{\dot\phi}\omega_{\beta 0} & k_\phi & 0 & -m'a'\omega_{\beta 0}^2\cos\beta \\ 0 & 0 & 0 & 0 & k & 0 \\ 0 & 0 & 0 & 0 & 0 & k' \end{bmatrix}$$

III. Case with Dampers

The passive AMCD spacecraft differential equations of motion with one damper and three symmetrical particles on both the AMCD rim and spacecraft are represented by the state equations,

$$\hat{M}(t)\ddot{x}(t)+\hat{B}(t)\dot{x}(t)+\hat{K}(t)x(t)=0 \tag{12}$$

where the matrices $\hat{M}$, $\hat{B}$, and $\hat{K}$ are as defined earlier. $\hat{M}(t)$ is an invertible matrix, so we can write the system (12) in the form

$$\ddot{x}(t)+B(t)\dot{x}(t)+K(t)x(t)=0 \tag{13}$$

where

$$B(t)=\hat{M}^{-1}(t)\hat{B}(t), K(t)=\hat{M}^{-1}\hat{K}(t)$$

To study the stability properties of the system (13), we represent it in the form

$$\dot{x}(t)=\begin{bmatrix} 0 & I \\ -K(t) & -B(t) \end{bmatrix} x(t);\ \dot{x}(t)=S(t)x(t) \tag{14}$$

Here $B(t)$ and $K(t)$ are continuous matrices of 6×6. In order to generate a Liapunov function, when $B(t)$ and $K(t)$ are not symmetrical, let us consider a function

$$V=-\langle XE,X\rangle \tag{15}$$

where we let

$$E^{-1}=\begin{bmatrix} \Gamma & -I \\ -I & \Lambda \end{bmatrix} \tag{16}$$

where matrices Γ and Λ are positive-definite constant matrices yet to be determined. I denotes identity matrices. E is positive definite, if and only if,

$$\Gamma>0, \qquad \Gamma\Lambda>I \tag{17}$$

The derivative of V along the trajectories of Eq. (15) yields:

$$\dot{V}=x^T E(E^{-1}S^T+SE^{-1})Ex \tag{18}$$

Therefore, V is a Liapunov function if a constant E^{-1} can be found such that

$$E^{-1}S^T(t)+S(t)E^{-1}\leq-\epsilon<0 \tag{19}$$

or

$$-(E^{-1}S^T(t)+S(t)E^{-1}) = \begin{bmatrix} 2I & \Gamma K^T-B^T-\Lambda \\ \Gamma K-B-\Lambda & B\Lambda+\Lambda B^T-K-K^T \end{bmatrix}$$

is semipositive definite if

$$(B\Lambda+\Lambda B^T-K-K^T)\geq\tfrac{1}{2}(\Gamma K-B-\Lambda)\times(\Gamma K^T-B^T-\Lambda) \tag{20}$$

Now we choose

$$\Gamma=\gamma I, \quad \Lambda=\lambda I \tag{21}$$

such that $\gamma\lambda>1$. The inequality (20) reduces to

$$\lambda(B+B^T)-(K+K^T)\geq\tfrac{1}{2}(\gamma K-B-\lambda I)\times(\gamma K^T-B^T-\lambda I) \tag{22}$$

For some positive constants b_1, b_2, and k_1, k_2 we can write the inequality (22) as

$$\{(b_2-b_1)^2I-[(2B-(b_1+b_2)I]\times[(2B-(b_1+b_2)I]^T\}+\gamma^2\{(k_2-k_1)^2I-[(2K-(k_1+k_2)I]\times[(2K-(k_1+k_2)I]^T\}$$
$$+\gamma(b_2-b_1)(k_2-k_1)\left\{\left[\frac{2B-(b_1+b_2)I}{(b_2-b_1)}+\frac{2K-(k_1+k_2)I}{(k_2-k_1)}+I\right]\times\left[\frac{2B-(b_1+b_2)I}{(b_2-b_1)}+\frac{2K-(k_1+k_2)I}{(k_2-k_1)}+I\right]^T\right\}$$
$$+\gamma(b_2-b_1)(k_2-k_1)\left\{\left[\frac{2B-(b_1+b_2)I}{(b_2-b_1)}\right]\times\left[\frac{2B-(b_1+b_2)I}{(b_2-b_1)}\right]^T+\left[\frac{2K-I(k_1+k_2)}{(k_2-k_1)}\right]\times\left[\frac{2K-I(k_1+k_2)}{(k_2-k_1)}\right]^T-I\right\}$$
$$+\{16k_1(\gamma b_1-1)-4(\lambda-b_1-\lambda k_1)^2\}+\{4(\lambda b_1+\gamma\lambda-2)-2\gamma^2(k_1+k_2)\}(K+K^T-2k_1I)$$
$$+\{4(\lambda+\gamma k_1)-2(b_1+b_2)\}(B+B^T-2b_1I)\geq 0 \tag{23}$$

Next, we examine the positiveness of each set of parentheses $\{\cdots\}$. We choose the constants $\gamma>0$, $\lambda>0$ such that $\gamma\lambda>1$ and satisfy the Lim and Kazda constant[3]

i)

$$b_i>\sqrt{k_2}-\sqrt{k_1} \tag{24}$$

ii)

$$b_2<b_1+\frac{4b_1+\sqrt{b_1^2-(\sqrt{k_2}-\sqrt{k_1})^2}}{(\sqrt{k_2}-\sqrt{k_1})^2}[k_1+\sqrt{k_1k_2}] \tag{25}$$

by setting the last three $\{\cdots\}$ equal to zero,

$$16k_1(\gamma b_1-1)-4(\lambda-b_1-\gamma k_1)^2=0 \tag{26}$$

$$4(\lambda b_1+\gamma\lambda-2)-2\gamma^2(k_1+k_2)=0 \tag{27}$$

$$4(\lambda+\gamma k_1)-2(b_1+b_2)=0 \tag{28}$$

From Eq. (27), we get

$$\lambda=2\gamma^{-1}-b_1+\tfrac{1}{2}\gamma(k_1+k_2) \tag{29}$$

On substituting λ in Eq. (26), we get

$$4\gamma^2k_1k_2-[4\gamma^{-1}+(k_1+k_2)\gamma-4b_1]^2=0$$

or

$$b_1=\gamma^{-1}+(\gamma/4)[\sqrt{k_2}-\sqrt{k_1}]^2 \tag{30}$$

or write the quadratic equation in γ as

$$\gamma^2(\sqrt{k_2}-\sqrt{k_1})^2-4b_1\gamma_1+4=0$$

Solving the quadratic equation in Eq. (30) yields:

$$\gamma=\frac{2(b_1+\sqrt{b_1^2-(\sqrt{k_2}-\sqrt{k_1})^2})}{(\sqrt{k_2}-\sqrt{k_1})^2} \tag{31}$$

From Lim and Kazda, constants $b_1>\sqrt{k_2}-\sqrt{k_1}$ implies that γ is real and positive. Now we solve Eq. (31) for b_2

$$b_1+b_2=2(\lambda+\gamma k_1)$$

On substituting λ from Eq. (29), we get

$$b_2=2(2\gamma^{-1}+\tfrac{1}{2}\gamma(k_1+k_2)-b_1)-b_1+2\gamma k_1 \tag{32}$$

From Eq. (30), we obtain b_1 as

$$2b_1=2\gamma^{-1}+\tfrac{1}{2}\gamma(k_1+k_2)-\gamma\sqrt{k_1k_2} \tag{33}$$

so that using the relation in Eq. (33) in Eq. (32) yields

$$b_2=2(2b_1+\gamma\sqrt{k_1k_2}-b_1)-b_1+2\gamma k_1$$
$$=b_1+2\gamma(k_1+\sqrt{k_1k_2})$$
$$=b_1+\frac{4(b_1+\sqrt{b_1^2-(\sqrt{k_2}-\sqrt{k_2})^2})}{(\sqrt{k_2}-\sqrt{k_1})^2}[k_1+\sqrt{k_1k_2}]$$

The last step follows by substituting γ from Eq. (31). From Eq. (30) we have

$$b_1=\gamma^{-1}+(\gamma/4)[\sqrt{k_2}-\sqrt{k_1}]^2$$

Substituting b_1 into Eq. (29) yields

$$\lambda=2\gamma^{-1}-(\gamma^{-1}+(\gamma/4)[\sqrt{k_2}-\sqrt{k_1}]^2)+\tfrac{1}{2}\gamma(k_1+k_2)$$
$$\lambda=\gamma^{-1}+(\gamma/4)(\sqrt{k_2}+\sqrt{k_1})^2 \tag{34}$$

Equations (31) and (34) give real constant positive γ and λ such that the Eqs. (26-28) are satisfied and the condition $\lambda\gamma>1$. Therefore, we have an autonomous Liapunov function whose time derivative is negative semidefinite. Using LaSalle's invariance principle, asymptotic stability can be concluded.

The results can be summarized into the following theorems:

Theorem 1: If there exist constants b_1, b_2 and k_1, k_2 satisfying the Lim and Kazda relation

i)

$$b_1>\sqrt{k_2}-\sqrt{k_1}$$

ii)

$$b_2<b_1+\frac{4b_1-\sqrt{b_1^2-(\sqrt{k_2}-\sqrt{k_1})^2}}{(\sqrt{k_2}-\sqrt{k_1})^2}[k_1+\sqrt{k_1k_2}]$$

and, in addition,

iii)

$$\left[\frac{2B-(b_1+b_2)I}{(b_2-b_1)}\right]\times\left[\frac{2B-(b_1+b_2)I}{(b_2-b_1)}\right]^T+\left[\frac{2K-(k_1+k_2)}{(k_2-k_1)}\right]\times\left[\frac{2K-(k_1+k_2)}{(k_2-k_1)}\right]^T\leq I$$

Then, the system Eq. (13) is stable in the sense of Liapunov.

Let the negative real part of $M[S(t)]$ denote the least negative real part of the eigenvalues of $S(t)$.

Theorem 2: Given a constant matrix A[a constant part of $S(t)$], there exists a positive matrix P such that
i)

$$\delta = \min \delta_1, \delta_2 \mid \delta_i = \min_{\alpha_i = 0-2\pi} M(S(t))$$

ii)

$M(B) \leq 0$ for all matrix $S(t)$ continuously with

$$\| S(t) - A \| < \delta \text{ for all } t \geq 0$$

$$B \stackrel{\text{def}}{=} A + (\delta/2) I$$

then the trivial solution of Eq. (13) is stable in the sense of Liapunov.

Proof: Since the real parts of the eigenvalues of

$$B \stackrel{\text{def}}{=} A - [\tfrac{1}{2}\delta + M(S(t))] I$$

are all negative or zero, then there exists a positive definite Hermitian matrix P such that

$$PB + B^T P = -C(C \geq 0)$$

The eigenvalues of

$$PS(t) + S^T(t)P + \delta P \stackrel{\text{def}}{=} L$$

vary continuously with $S(t)$ and as $S(t) \to A$, the matrix L tends to $PA + A^T P - \{\delta + 2M[S(t)]\} P = PB + B^T P = -C$. Therefore the matrix L is seminegative definite throughout some neighborhood $\| S(t) - A \| < \delta$ of A.

Now define a Liapunov function

$$V(X) = X^T PX; \; P = P^T > 0$$

then $\dot{V}$ along the trajectories of Eq. (14) yields:

$$\dot{V}(X) = X^T [PS(t) + S^T(t)P] X \leq X^T [\delta + 2M(S(t))] PX$$

$$= -\delta X^T PX = -\delta V < 0$$

The rest of the arguments are standard and so omitted. This completes the proof.

The prototype AMCD spacecraft was found to be asymptotically stable where the parameters used are: $I_a = 680$ kg − m^2, $I_{az} = 1360$ kg − m^2, $I_s = 680$ kg − m^2, $I_{sz} = 453.3$ kg − m^2, $a = 0.76$ m, $a' = 0.76$ m, $\omega_{\alpha 0} = 401.3$ rad/sec, $\omega_{\beta 0} = 0.1$ rad/sec, $k_\phi = 1020$, $k_{\dot\phi} = 2856$. Simulation was reported for larger k_ϕ, $k_{\dot\phi}$, where the system was found stable. A considerable amount of work is needed for selection of the best parameters. Further simulation is needed when one or more ball-in-tube dampers are present. Note that smaller damper masses appear to stabilize the system.

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Fourth-Order Runge-Kutta Integration with Stepsize Control

David G. Hull*
University of Texas at Austin, Austin, Texas

Introduction

AN engineer wanting to integrate a system of ordinary differential equations has at his disposal a wide variety of numerical integration methods. Included are single-step (Runge-Kutta) integrators up to eighth order[1,2] and multistep (Adams-Bashforth, Adams-Moulton) integrators of any order.[3] Furthermore, these integrators are available in both fixed-step and variable-step versions. In spite of this, however, the fixed-step, fourth-order Runge-Kutta method is used with a degree of regularity even though the problem being solved does not require fixed-step integration. Perhaps the reason for this is that engineers tend to use what has worked in the past and do not always have time to experiment with numerical methods.

It is well known that variable-step integration is more efficient than fixed-step integration; that is, fewer integration steps are required to achieve the same accuracy. Hence, the purpose of this Note is to present a simple step-size control procedure for the classical fourth-order Runge-Kutta method. Since the implementation of the step-size procedure requires only about 15 FORTRAN statements, anyone using the fixed-step version can make the conversion quite easily.

Step-size control already is available for the fourth-order Runge-Kutta method.[4] One method, doubling, uses two regular steps and a simultaneous double step to estimate the truncation error of the fourth-order method. It requires five and one-half function evaluations per integration step and costs seven function evaluations for a rejected step. Another method uses a sequence of accepted steps to predict the next step. However, this approach requires additional storage, and the single-step character of the Runge-Kutta method is destroyed.

The step-size control procedure being proposed here uses the concept of embedded methods. Here, a lower-order method using the same function evaluations as the fourth-order method is isolated. Because of the nature of the function evaluations needed to generate the fourth-order method, a third-order method cannot be found, and a second-order method must be used, as indicated by Sarafyan (Ref. 5, p. 71). The difference of the solutions obtained by the second- and fourth-order methods is used as an estimate of the truncation error of the second-order method. This approximate truncation error then is used to find the size of the integration step which maintains a prescribed relative error.

Fixed-Step Integration

The fourth-order Runge-Kutta method is to approximate the solution of the initial-value problem

$$\frac{dx}{dt} = f(t,x), \quad x_0 = x(t_0) \tag{1}$$

by the relation

$$\hat{x}(t_0 + h) = x_0 + h \sum_{k=0}^{3} \hat{c}_k f_k \tag{2}$$

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*Associate Professor, Department of Aerospace Engineering and Engineering Mechanics. Member AIAA.